

Cosmological Vorticity Perturbations, Gravitomagnetism, and Mach's Principle

CHRISTOPH SCHMID

*Theoretische Physik, ETH-Zürich, CH-8093 Zürich
Switzerland*

The axes of gyroscopes experimentally define non-rotating frames. But what physical cause governs the time-evolution of gyroscope axes ? Starting from an unperturbed, spatially flat FRW cosmology, we consider cosmological vorticity perturbations (i.e. vector perturbations, rotational perturbations) at the linear level. We ask: Will cosmological rotational perturbations drag the axis of a gyroscope relative to the directions (geodesics) to galaxies beyond the rotational perturbation? We cast the laws of Gravitomagnetism into a form showing clearly the close correspondence with the laws of ordinary magnetism. Our results are:

- 1) The dragging of a gyroscope axis by rotational perturbations beyond the \dot{H} radius (H = Hubble constant) is exponentially suppressed.
- 2) If the perturbation is a homogeneous rotation inside a radius significantly larger than the \dot{H} radius, then the dragging of the gyroscope axis by the rotational perturbation is exact for any equation of state for cosmological matter.
- 3) The time-evolution of a gyroscope axis exactly follows a specific average of the matter inside the \dot{H} radius for any equation of state.

In this precise sense Mach's Principle follows from cosmology with Einstein Gravity.

PRESENTED AT

COSMO-01

Rovaniemi, Finland,
August 29 – September 4, 2001

1 Introduction and Conclusions

The spin axes of gyroscopes define our local nonrotating frame experimentally. But what physical cause governs the time-evolution of gyroscope axes? Newton invoked “absolute space”. Mach [1] made the hypothesis that the nonrotating frame is determined (caused) “in some way” by the motions of all masses in the universe. He did not know any mechanism, any force, which would have this effect.

We shall show that General Relativity, specifically Gravitomagnetism, in the context of the Friedmann-Robertson-Walker (FRW) cosmology, spatially flat, with rotational perturbations (treated as linear perturbations) makes predictions in total agreement with Mach’s hypothesis as formulated above.

Thirring in 1918 [2] analyzed the partial dragging of inertial frames inside a rotating infinitely thin spherical shell of uniform surface mass density and total mass M . He found that general relativity in the weak field approximation and in first order in the angular velocity Ω_{shell} predicts an acceleration of test particles in the interior corresponding to a Coriolis force, which could be eliminated by going to a reference frame which is rotating with $\Omega' = f_{\text{drag}}\Omega_{\text{shell}}$. In the weak field approximation, $G_N M \ll R$, he obtained the dragging fraction $f_{\text{drag}} = \frac{4}{3} \frac{G_N M}{R} \ll 1$. Lense and Thirring [2] made the corresponding analysis outside a rotating star.

Einstein, after initially being inspired by Mach’s ideas, concluded in 1949 [3]: “Mach conjectures that inertia would have to depend upon the interaction of masses, precisely as was true for Newton’s other forces, a conception which for a long time I considered as in principle the correct one. It presupposes implicitly, however, that the basic theory should be of the general type of Newton’s mechanics : masses and their interactions as the original concepts. The attempt at such a solution does not fit into a consistent field theory, as will be immediately recognized.” We shall show in section 4, how this apparent contradiction with Mach’s conjecture is resolved in General Relativity, specifically in Gravitomagnetism.

Goedel in 1949 [4] presented a model which contradicts Mach’s principle, but since it contains closed time-like curves, it is not relevant for physics.

Brill and Cohen in 1966 [5] again considered a very thin rotating spherical shell to lowest order in the rotation frequency, but now to all orders in the mass M of the shell, i.e. they treated geometry as a perturbation of Schwarzschild geometry. They found that space-time is flat throughout the interior and that the interior Minkowski space is rotating relative to the asymptotic Minkowski space. For large masses, whose Schwarzschild radius approaches the shell radius, Brill and Cohen found that the induced rotation inside approaches the rotation of the shell, i.e. the dragging of the inertial frame inside by the rotating matter of the shell becomes perfect.

A cosmologically more relevant example is a rotating ball of dust and a gyroscope at the center. In a weak-field approximation with a Minkowski background this is the same as a superposition of a sequence of Thirring shells, which gives f_{drag} of the order of $G_N M_{\text{ball}}/R_{\text{ball}}$. The Friedman equation gives $G_N \rho$ of order of $H^2 \equiv R_H^{-2}$, if the curvature

term is not dominant. Hence f_{drag} is of order of R_{ball}^2/R_H^2 , valid for $R_{\text{ball}} \ll R_H$. Hence the dragging factor approaches the order of 1 only for $R_{\text{ball}} \rightarrow R_H$, but in this case the weak field approximation on a Minkowski background breaks down. See also the discussion of Mach's principle in Misner, Thorne, and Wheeler [6].

C. Klein [7] analyzed a thin spherical shell with empty interior embedded in a Friedmann universe. The shell is required to follow the expanding motion of the surrounding cosmic dust. Rotations are treated to first order in the angular velocity. In his treatment the dragging coefficient tends to one, if nearly "the whole mass of the universe" is concentrated in the thin shell.

At a conference on Mach's Principle in 1993, edited by Barbour and Pfister [8], the question "Is general relativity with appropriate boundary conditions of closure of some kind perfectly Machian?" was put to the participants at the end of the conference. The vote was "No" with a clear majority (page 106 of [8]).

In this paper we discuss a realistic cosmological model and use cosmological perturbation theory including super-horizon perturbations (instead of perturbations around Minkowski space or around the Schwarzschild solution), we use realistic cosmological matter (instead of matter with a contrived energy-momentum-stress tensor), and we analyze the most general cosmological perturbations in the vorticity sector (instead of toy models like rigidly rotating thin shells). We consider a background FRW cosmology, spatially flat, with linear vorticity perturbations, and we obtain the following specific results:

- 1) The dragging of gyroscope axes by rotational perturbations beyond their \dot{H} radius is exponentially suppressed, where H = Hubble constant.
- 2) For a homogeneous rotation of cosmological matter inside a perturbation radius R_p , the dragging of the axis of a gyroscope at the center approaches exact dragging exponentially fast as R_p increases beyond the \dot{H} radius. This holds for any equation of state for cosmological matter.
- 3) A gyroscope axis exactly follows a specific average of the energy flow in the universe with an exponential cutoff outside the \dot{H} radius for any equation of state.

In this precise sense we have shown that for a spatially flat FRW Universe with rotational perturbations (added at the linear level) Mach's Principle on nonrotating frames follows from General Relativity (without the need to impose any boundary condition of closure).

2 Vorticity Perturbations

The vorticity $\vec{\omega}$ of a velocity field \vec{v} is defined by $\vec{\omega} = \text{curl } \vec{v}$. Every velocity field can be decomposed into a potential flow, $\text{curl } \vec{v} = 0$, and a vorticity flow, $\text{div } \vec{v} = 0$. The simplest vorticity perturbation is a spherical region which rotates with a homogeneous

angular velocity Ω around an axis through its center, $\vec{v} = \Omega \vec{e} \times \vec{r}$. A more realistic cosmological perturbation is one with a Gaussian cutoff, $\Omega(r) = \Omega_0 \exp(-r^2/R_p^2)$. In the cosmological context a perturbation in a spherical region with a velocity field regular at the origin is much more appropriate than the vorticity field familiar from hydrodynamics with cylindrical symmetry and a vortex-line singularity in the center.

The most general vorticity field \vec{v} can be given by an expansion in vector spherical harmonics $\vec{X}_{\ell,m}(\theta, \phi)$, ref. [9]:

$$\vec{X}_{\ell,m}(\theta, \phi) = \vec{L} Y_{\ell,m}(\theta, \phi), \quad \vec{L} = -i \vec{r} \times \vec{\nabla}, \quad (1)$$

$$\vec{v}(\vec{r}) = \sum_{\ell,m} A_{\ell,m}(r) \vec{X}_{\ell,m}(\theta, \phi) + \sum_{\ell,m} \text{curl} \left[B_{\ell,m}(r) \vec{X}_{\ell,m}(\theta, \phi) \right]. \quad (2)$$

The simplest case is $\ell = 1$, which can be oriented such that $m = 0$: The A -solution is a homogeneous rotation at each value of r (see the simple examples above), which is called a “toroidal” field. The B -solution is called a “poloidal field”. If the velocity field is toroidal, the vorticity field is poloidal and vice versa. These two configurations are familiar from ordinary magnetostatics, where $\text{curl } \vec{B} = 4\pi \vec{J}$.

3 Cosmological Perturbation Theory

For superhorizon perturbations we need a general relativistic treatment. This was developed by J. Bardeen in his Phys. Rev. article of 1980 (ref. [10]). In linear perturbation theory one has three decoupled sectors of perturbations: scalar (density perturbations), vector (vorticity or rotational perturbations), tensor (gravitational wave perturbations) under rotations in 3-space. In the vector sector all perturbation quantities must be built from 3-vector fields with zero divergence, i.e. from pure vorticity fields.

The perturbation δg_{00} is a scalar under 3-rotations, hence

$$\delta g_{00} = 0. \quad (3)$$

Therefore the slicing of space-time by space-like hypersurfaces of fixed time Σ_t has measured times between slices for $\Delta t = 1$ (lapse) unperturbed.

In contrast it is not useful (although possible) to insist on a time-orthogonal foliation (hence time-orthogonal coordinates). Rather we allow that the lines $\vec{x} = \text{const}$ are not orthogonal on the hypersurfaces of constant time Σ_t . It is useful to introduce the concept of a field of fiducial observers (FIDO’s), which are located at fixed values of \vec{x} , ref. [11]. The shift-3-vector field $\vec{\beta}$ is defined as the 3-velocity $d\vec{v}/dt$ of the FIDO relative to the normals on Σ_t .

One can easily show [10] that it is always possible to find a gauge transformation (coordinate transformation) such that the new coordinates give

$$\delta^{(3)} g_{ij} = 0. \quad (4)$$

Therefore the 3-geometry of Σ_t is unperturbed. We take the FRW background spatially flat, hence Σ_t has the geometry of the Euclidean 3-space. In the gauge of eq. (4) Σ_t has comoving Cartesian coordinates,

$$ds^2 = -dt^2 + a^2(t)(dx^i)^2 + 2a\beta_i dx^i dt . \quad (5)$$

Geodesics on Σ_t are straight lines on our choice of chart. In all other gauges the coordinates of Euclidean 3-space are such that if the coordinates are Cartesian at one time, at all other times the coordinate lines (e.g. the x-axis) are no longer geodesics, they are getting wound up relative to geodesics.

Asymptotically (beyond the region of the perturbation) we have FRW geometry and FRW coordinates, hence distant galaxies (galaxies beyond the rotational perturbation) are at fixed \vec{x} (comoving coordinates). Therefore the coordinate axes in our gauge stay fixed relative to distant galaxies, i.e. the coordinate axes are geodesics on Σ_t and nonrotating relative to distant galaxies. In the context of rotating black holes such coordinates are called “star-fixed coordinates”. For brevity of notation we shall employ this short term, although we always mean “coordinates fixed to galaxies beyond the rotational perturbation”.

4 The Laws of Gravitomagnetism for “star”-fixed FIDO’s

By what dynamical mechanism can cosmological matter in rotational motion far away influence the spin axes of gyroscopes here? By the laws of general relativity, more precisely gravitomagnetism. Since we work to first order in vorticity perturbations, we must analyze weak-field gravitomagnetism. We cast the laws of weak-field gravitomagnetism into a form which shows clearly the close correspondence with the familiar laws of ordinary magnetism (electromagnetism). We introduce and work with the gravitomagnetic field \vec{B}_g and the gravitoelectric field \vec{E}_g measured by fiducial observers (FIDO’s) ref. [11], where the FIDO is a crucial concept for our formulation of the laws of gravitomagnetism. Note that for observers who are free-falling and non-rotating relative to gyroscopes, i.e. for inertial observers, there are no gravitational forces, $\vec{E}_g = 0$, $\vec{B}_g = 0$, by the equivalence principle.

Our choice is to work with “star”-fixed FIDO’s (more precisely FIDO’s fixed by galaxies beyond the rotational perturbation). “Star”-fixed FIDO’s are defined to stay at fixed \vec{x} in the “star”-fixed coordinate system, and (as in general) $\bar{e}_0(\text{FIDO}) = \bar{u}(\text{FIDO})$. Furthermore the spatial unit vectors \bar{e}_j of the FIDO’s LONB (local orthonormal basis) point towards fixed distant galaxies.

The 3-momentum measured by FIDO’s is denoted by $\vec{p} = p^{\hat{j}} \bar{e}_{\hat{j}}$. Hats on indices refer to LONB’s, the components $p^{\hat{j}}$ are quantities directly measured by FIDO’s. Note that this 3-vector lies in the hyperplane orthogonal to $\bar{u}(\text{FIDO})$, not in the hyperplane Σ_t .

The time-derivative $d\vec{p}/dt$ measured by FIDO's for a free-falling test particle gives the operational definition of the gravitational force,

$$(d\vec{p}/dt)_{\text{free fall}} =: \vec{F}_g . \quad (6)$$

The gravitoelectric field \vec{E}_g is operationally defined via quasistatic test particles,

$$\vec{E}_g := \left(\frac{\vec{F}_g}{m} \right)_{\text{quasistatic particle}} . \quad (7)$$

$\vec{E}_g = \vec{g}$ is the gravitational acceleration relative to the FIDO (i.e. measured by a FIDO) for a free-falling particle. Compare with FIDO's on the surface of the earth (at fixed r), who measure $\vec{g} = -9.8 \text{ m/s}^2 \vec{e}_r$. The gravitomagnetic field \vec{B}_g is defined via the first-order term in the velocity of the test particle (we put $c = 1$),

$$(\vec{F}_g)_{1^{\text{st}} \text{ order in } v} = m(\vec{v} \times \vec{B}_g) . \quad (8)$$

After having given the operational definitions of \vec{E}_g and \vec{B}_g , which are valid for any choice of FIDO's, and having fixed our choice of FIDO's ("star"-fixed), we can derive the laws of gravitomagnetism. The results are:

- 1) Equation of motion for test particles. We give it for vorticity perturbations of Minkowski space (for simplicity in this conference report),

$$d\vec{p}/dt = \varepsilon \left(\vec{E}_g + \vec{v} \times \vec{B}_g \right) , \quad (9)$$

where ε is the energy measured by the FIDO. This law is valid for free-falling particles of arbitrary velocities (e.g. photons), but only for "star"-fixed FIDO's. For other FIDO's there will be terms bilinear in the velocity. Our law should be compared to the Lorentz law of electromagnetism $d\vec{p}/dt = q(\vec{E} + \vec{v} \times \vec{B})$.

- 2) Equation of motion for the spin of a test particle or gyroscope at rest with respect to the FIDO.

$$d\vec{S}/dt = -\frac{1}{2} \vec{B}_g \times \vec{S} . \quad (10)$$

This gives an angular velocity of precession of the spin axis relative to the axes of the FIDO (which in our case is relative to the direction to distant galaxies),

$$\vec{\Omega}_{\text{precession}} = -\frac{1}{2} \vec{B}_g . \quad (11)$$

This should be compared to the analogous precession of the intrinsic angular momentum (spin) of classical charged particles in an electromagnetic field, $\vec{\Omega}_{\text{precession}} =$

$+\frac{1}{2} \frac{q}{m} \vec{B}$. Note again that for an inertial observer $d\vec{p}/dt = 0$, $d\vec{S}/dt = 0$. Everything depends on the choice of the field of FIDO's. Using eqs. (8) and (11) we obtain $\vec{F}_g = 2m(\vec{\Omega}_{\text{precession}} \times \vec{v})$, i.e. the Coriolis force, where we must remember that $\vec{\Omega}_{\text{precession}}$ is minus the rotation velocity of the FIDO relative to the gyroscope axis. Since we are working to lowest order in $\vec{\Omega}$ the centrifugal term vanishes. A homogeneous gravitomagnetic field can be transformed away completely by going to a rigidly rotating coordinate system, i.e. physics on a merry-go-round in Minkowski space is equivalent to physics in a homogeneous gravitomagnetic field. This is analogous to the equivalence between physics in a homogeneous Newtonian gravitational field and physics in a linearly accelerated frame.

- 3) Relation between \vec{E}_g , \vec{B}_g and the shift vector $\vec{\beta}$. The gravitational metric perturbations $\delta g_{\mu\nu}$, i.e. the gravitational potentials for vorticity perturbations in “star”-fixed coordinates are given by the shift vector $\vec{\beta}$, which is connected to \vec{E}_g and \vec{B}_g by

$$\vec{E}_g = -\partial_t \vec{\beta}, \quad \vec{B}_g = \vec{\nabla} \times \vec{\beta}. \quad (12)$$

We see that the shift vector $\vec{\beta}$ plays exactly the same role for gravitomagnetism as the electromagnetic vector potential \vec{A} for electromagnetism. Therefore we can write $\vec{\beta} = \vec{A}_g$, and call it the gravitomagnetic vector potential.

- 4) Einstein's $G_{\hat{0}\hat{j}}$ equation for weak-field gravitomagnetism:

A) In Minkowski background

$$\nabla \times \vec{B}_g = -16 \pi G_N \vec{J}_\epsilon, \quad (13)$$

where \vec{J}_ϵ is the energy current, and in linear perturbation theory $\vec{J}_\epsilon = (\rho + p)\vec{v}$ with $\vec{v} = \frac{d\vec{x}}{dt}$. Compare Einstein's $G_{\hat{0}\hat{j}}$ equation to the Ampère-Maxwell equation in electromagnetism, $\nabla \times \vec{B} - \partial_t \vec{E} = +4\pi \vec{J}_q$, where \vec{J}_q is the charge current. In weak-field gravitomagnetism for “star”-fixed FIDO's the $G_{\hat{0}\hat{j}}$ equation is a constraint equation (i.e. no time derivatives, an equation at one moment of time with the same form as Ampère's law of magnetostatics), while the Ampère-Maxwell equation contains a time-derivative and is not a constraint equation.

B) In a FRW background eq. (13) gets an extra term,

$$\nabla \times \vec{B}_g - 4\dot{H} \vec{\beta} = -16 \pi G_N \vec{J}_\epsilon. \quad (14)$$

From $\dot{H} = -4\pi G_N(\rho + p)$ we see that $\dot{H} \leq 0$ for $p \geq -\rho$, therefore we define the \dot{H} radius by $R_H^2 = (-\dot{H})^{-1}$, and we define $\mu^2 = -4\dot{H} = 4R_H^{-2}$. We insert the vector potential $\vec{A}_g = \vec{\beta}$, we use $\text{div } \vec{A}_g = 0$ and $\text{curl curl } \vec{A}_g = -\Delta \vec{A}_g$, hence we obtain by Fourier transformation

$$(k_{\text{phys}}^2 + \mu^2) \vec{A}_g = -16 \pi G_N \vec{J}_\epsilon. \quad (15)$$

We see that the new term on the left-hand side, $(-4\dot{H} \vec{\beta})$ resp $(\mu^2 \vec{A}_g)$, dominates for superhorizon perturbations. The solution is the Yukawa potential for the source \vec{J}_ε ,

$$\vec{A}_g(\vec{x}, t) = \vec{\beta}(\vec{r}, t) = -4 G_N \int d^3 r' \vec{J}_\varepsilon(\vec{r}', t) \frac{\exp(-\mu|\vec{r} - \vec{r}'|)}{|\vec{r} - \vec{r}'|}, \quad (16)$$

analogous to the formula for ordinary magnetostatics except for the exponential cutoff.

5 Mach's Principle

The Yukawa potential in eq. (16) has an exponential cutoff for $|\vec{r} - \vec{r}'| \geq \frac{1}{\mu}$, hence we reach our first important conclusion: The contributions of vorticity perturbations beyond the \dot{H} radius are exponentially suppressed.

Our second result concerns the exact dragging of gyroscope axes by a homogeneous rotation of cosmological matter out to significantly beyond the \dot{H} radius (for the exponential cutoff to be effective). This holds for any equation of state. This is easily seen from Einstein's G_{0j} equation in k -space for superhorizon perturbations, $k_{\text{phys}}^2 \ll (-\dot{H})$,

$$-4\dot{H}\vec{\beta} = -16\pi G_N \vec{J}_\varepsilon, \quad \vec{J}_\varepsilon = (\rho + p) \vec{v}_{\text{fluid}}, \quad \dot{H} = -4\pi G_N(\rho + p). \quad (17)$$

All the prefactors cancel, and we obtain $\vec{\beta}(\vec{x}) = -\vec{v}_{\text{fluid}}(\vec{x})$. With $\vec{\Omega}_{\text{gyroscope}} = -\frac{1}{2}\vec{B} = -\frac{1}{2}(\vec{\nabla} \times \vec{\beta})$ from eqs. (11) and (12), and with $\vec{\Omega}_{\text{fluid}} = \frac{1}{2}(\vec{\nabla} \times \vec{v}_{\text{fluid}})$ we obtain

$$\vec{\Omega}_{\text{gyroscope}} = \vec{\Omega}_{\text{fluid}}. \quad (18)$$

This proves exact dragging of gyroscope axes here by a homogeneous rotation of cosmological matter out to significantly beyond the \dot{H} radius.

Our third result concerns the most general vorticity perturbation in linear approximation, and it states what specific average of energy flow in the universe determines the motion of gyroscope axes here at $r = 0$,

$$\begin{aligned} \vec{\Omega}_{\text{gyroscope}}(r = 0, t) &= -\frac{1}{2} \vec{B}_g(r = 0, t) = \\ &2G_N(\rho + p) \int d^3 r \frac{1}{r^3} \left[(1 + \mu r) e^{-\mu r} \right] [\vec{r} \times \vec{v}(\vec{r}, t)]. \end{aligned} \quad (19)$$

The right-hand side of this equation is the gravitomagnetic moment of the energy current distribution on a shell $[r, r + dr]$, analogous to the magnetic moment of an electric current distribution, integrated over r with the given weight function. This is the lowest term, the $l = 1$ term, in the multipole expansion for $r_{\text{obs}} = 0$ and $r_{\text{source}} > 0$. Higher multipoles do not contribute to $\vec{\Omega}_{\text{gyroscope}}$ at $r = 0$. Only the toroidal velocity field for $l = 1$ of eq. (1)

contributes in eq. (19), i.e. a term corresponding to a rigid rotation velocity $\vec{\Omega}(r)$ for each shell $(r, r + dr)$.

Eqs. (16) and (19) are equations at fixed time, the precession of a gyroscope axis here is determined by the rotational motion of the masses in the universe at the same cosmic time. This shows that Einstein's objection to Mach's principle of 1949 [3] is not valid for weak gravitomagnetism. Furthermore, because of the exponential cutoff, which follows from Einstein's equations in the cosmological context, there is no need to impose "an appropriate boundary condition of some kind".

Is exact dragging a measurable effect in principle? There seem to be two problems: 1) To obtain exact dragging (within some given experimental error) the observational cutoff radius must be significantly larger than the \dot{H} radius (for the exponential cutoff to be effective). We have defined $\vec{\Omega}_{\text{gyroscope}}$ and $\vec{\Omega}_{\text{fluid}}$ relative to distant galaxies, i.e. galaxies beyond the rotational perturbation. But we cannot "see" galaxies beyond the \dot{H} radius. 2) The G_{0j} equation for "star"-fixed FIDO's is an equation at fixed time, our results hold at fixed time, but an experimental test today with telescopes here on earth cannot see the galaxies at the same cosmic time, it sees galaxies on the past light cone. The solution to these problems is an experimental test in principle: Take data locally by many different observers all over \vec{x} space, also beyond the \dot{H} radius. Collect and patch together the data after a few Hubble times. This can be done as a computer experiment, where the resulting test only involves our measurable quantities. This computer experiment is not necessary, since we have given the proof (at the level of linear perturbation theory).

We conclude that in a spatially flat FRW Universe with rotational perturbations (treated at the linear level) Mach's Principle on nonrotating frames follows from General Relativity.

ACKNOWLEDGEMENTS

We thank T. Cereghetti for collaboration while he was working on his diploma thesis at ETH.

References

- [1] E. Mach, *Die Mechanik in Ihrer Entwicklung Historisch-Kritisch Dargestellt* (Brockhaus, Leipzig, 1912). English translation by T.J. McCormack, *The Science of Mechanics: A Critical and Historical Account of Its Development*, (6th ed., Open Court Publishing, La Salle, Ill., 1960).
- [2] H. Thirring, Phys. Zeitschr. **19**, (1918) 33, J. Lense and H. Thirring, Phys. Zeitschr. **19**, (1918) 156.

- [3] A. Einstein, “Autobiographical Notes”. In *Albert Einstein, Philosopher-Scientist*, ed. P.A. Schlipp, Evanston, Illinois: The Library of Living Philosophers, Inc., Illinois (1949) 29.
- [4] K. Goedel, Rev. Mod. Phys. **21**, (1949) 447.
- [5] D.R. Brill and J.M. Cohen, Phys. Rev. **143**, (1966) 1011, J.M. Cohen and D.R. Brill, Nuovo Cim. **54**, (1968) 209.
- [6] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation*, Freeman (1973).
- [7] C. Klein, Classical and Quantum Gravity **10**, 1619 (1993).
- [8] J.B. Barbour and H. Pfister, *Mach’s Principle: From Newton’s bucket to quantum gravity* (Birkhäuser, 1995).
- [9] J.D. Jackson, *Classical Electrodynamics*, Wiley (1999).
- [10] J.M. Bardeen, Phys. Rev. D **22**, (1980) 1882.
- [11] K.S. Thorne, R H. Price, and D.A. Macdonald, editors, *Black Holes, The Membrane Paradigm*, Yale University Press (1986).